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2d - PLATE MODELS OBTAINED FROM 3d - ELASTICITY MODELS

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1. STATEMENT OF THE PROBLEM ; NOTATION

Summation convention ; dx-symbols omitted in $\int$

Latin indices : $i, j, p, \ldots \in \{1,2,3\}$

Greek indices : $\alpha, \beta, \gamma, \ldots \in \{1,2\}$

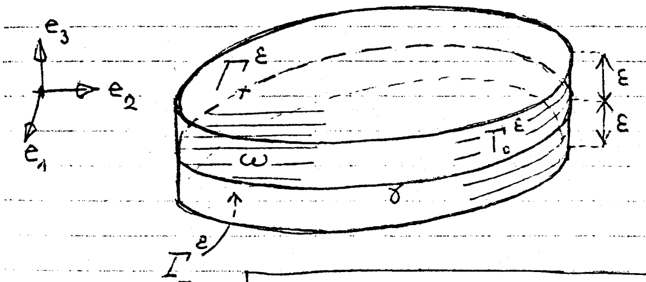
$\partial_i v = \frac{\partial}{\partial x_i}$, $\partial_{ij} v = \frac{\partial^2}{\partial x_i \partial x_j}$.

1.1. The clamped plate problem ; the linear case.

$\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$; Applied forces: $f = (f_i)$ in Ω^ε

$g = (g_i)$ on $\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon$

$u = (u_i) = 0$ on Γ_0^ε .



(1)
$$\boxed{\begin{aligned} &J(u) = \inf_{v \in V^\varepsilon} J(v) \quad , \quad V^\varepsilon = \{ v \in (H^1(\Omega^\varepsilon))^3 ; v = 0 \text{ on } \Gamma_0^\varepsilon \} , \\ &J(v) = \frac{1}{2} \int_{\Omega^\varepsilon} (A^{-1}\gamma(v))_{ij} \gamma_{ij}(v) - \left\{ \int_{\Omega^\varepsilon} f_i v_i + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i v_i \right\} , \end{aligned}}$$

$$\boxed{\gamma_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)} ,$$

Young's modulus, Poisson's coef.

$$\boxed{(AX)_{ij} = \left(\frac{1+\nu}{E}\right) X_{ij} - \frac{\nu}{E} X_{pp} \delta_{ij}} \qquad (E > 0, \ 0 < \nu < \tfrac{1}{2}) ,$$

$$(A^{-1}Y)_{ij} = \left(\frac{E}{1+\nu}\right) Y_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)} Y_{pp} \delta_{ij} . \quad \text{(Lamé's constants)}$$

Equivalent system (obtained from the variational equations $J'(u)v = 0$ for all $v \in V^\varepsilon$)

$$(2) \qquad \boxed{\begin{aligned} -\partial_j (A^{-1}\gamma(u))_{ij} &= f_i \text{ in } \Omega^\varepsilon \\ u &= 0 \text{ on } \Gamma_0^\varepsilon \\ (A^{-1}\gamma(u))_{i3} &= \pm g_i \text{ on } \Gamma_\pm^\varepsilon \; (1) \end{aligned}}$$

When ε is "small", people solve instead the well-known biharmonic problem (assuming $f_\alpha = g_\alpha = 0$ for convenience):

$$(3) \qquad \boxed{\begin{aligned} &\frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta^2 u_3 = f \text{ in } \omega \; \left(f \overset{def}{=} g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3 \, dx_3 \right) \\ &u_3 = \partial_\nu u_3 = 0 \text{ on } \gamma \end{aligned}}$$

Questions: - How do we go from (2) to (3)? (In books of Mechanics, e.g. Landau & Lifchitz ∦, this is achieved through a priori assumptions, geometrical or mechanical in nature).

- In particular, how a system "degenerates in a single equation?; how a 2nd-order problem becomes a 4th-order problem?; how do we obtain the boundary conditions $u_3 = \partial_\nu u_3 = 0$ (the "clamped" plate problem)?

(1) Special case of the general b.c. $(A^{-1}\gamma(u))_{ij}\nu_j = g_i$

One mathematical way to answer these questions is the following: (1)

(i) The problem is written in the mixed form

(4) $$\boxed{\begin{aligned} (A\sigma)_{ij} &= \gamma_{ij}(u) \\ -\partial_j \sigma_{ij} &= f_i \\ u &= 0 \text{ on } \Gamma_0^\varepsilon \\ \sigma_{i3} &= \pm g_i \text{ on } \Gamma_\pm^\varepsilon, \end{aligned}} \qquad (\Leftrightarrow \sigma_{ij} = (A^{-1}\gamma(u))_{ij})$$

i.e., the unknowns are not only the u_i's but also the σ_{ij}'s ($\sigma = (\sigma_{ij})$ = stress tensor). In variational form, these equations represent the Hellinger-Reissner variational principle.

Remark: Using the stress-displacement formulation rather than the displacement formulation is crucial for the success of the method.

(ii) Pose the problem over a set Ω $(= \omega \times]-1,1[)$ independent of ε, and apply the asymptotic expansion method (2) | cf. especially J.L. LIONS, Lecture Notes in Math. vol. 323, Springer, for problems posed in variational form

$$\begin{cases} u^\varepsilon = \varepsilon^p u^p + \varepsilon^{p+1} u^{p+1} + \dots & \text{(for an appropriate } p \in \mathbb{Z}) \\ \sigma^\varepsilon = \varepsilon^p \sigma^p + \varepsilon^{p+1} \sigma^{p+1} + \dots \end{cases}$$

(iii) Then: – we find that u_3^p is precisely the solution of (3) (after returning to the set Ω^ε);

– we can estimate $\|u^\varepsilon - \varepsilon^p u^p\|$ in appropriate norms (cf. a forthcoming joint paper and

(2) See K.O. FRIEDRICHS, ~~see~~ A.L. GOL'DENVEIZER for the application of the a.e.m. to equations (rather than var. eqns), with simplifying assumpt. and no error anal.

(1) See P.G. CIARLET and P. DESTUYNDER: "A justification of the two-dimensional linear plate model" (to appear).

Destuynder's thesis).

<u>Comments</u>: The computation of $u^{\Gamma+1}$ involves a <u>boundary layer phenomenon</u> (in this sense, it is a <u>singular perturbation problem</u>); cf. Destuynder's thesis.

We can analyze similarly the eigenvalue problem ([1]), and shell problems (cf. Destuynder's thesis).

The above considerations will be made more specific in the nonlinear case (cf. Sect. 3-4).

1.2 • <u>The nonlinear case</u>[2] The 3d-model will be described in a moment; the 2d-model we have in mind is the famed <u>von Kármán equations</u>:

(5)
$$\boxed{\begin{aligned} a\,\Delta^2 u_3 &= [\psi, u_3] + f, \\ b\,\Delta^2\psi &= -[u_3, u_3], \end{aligned}}$$
← as found for instance in LIONS' book; cf. BREZZI, MIYOSHI

where $a, b > 0$,

$$[f, g] = \partial_{11} f\, \partial_{22} g + \partial_{22} f\, \partial_{11} g - 2\partial_{12} f\, \partial_{12} g,$$

ψ is the <u>Airy stress function</u>, from which one may computes the functions $\sigma^0_{\alpha\beta} = \sigma_{\alpha\beta}(\cdot, \cdot, 0)$.

<u>Remark</u>: instead of the form $a\Delta^2 u_3 = [\psi, u_3] + f$, one finds also

(5')
$$\boxed{a\,\Delta^2 u_3 = [\psi, u_3] + [\Phi_0, u_3]},$$

(2) cf. P.G. CIARLET and P. DESTUYNDER: A justification of a nonlinear model in plate theory, to appear in Computer Methods in Applied Mechanics and Engineering (Proc. FENOMECH'78, Stuttgart).

(1) cf. P.G. CIARLET and S. KESAVAN (to appear).

for instance in (1)

This difference is one of the points we wish to clarify (among others things)

Boundary conditions:

(6) $\boxed{u_3 = \partial_\nu u_3 = 0 \quad \text{on } \gamma}$ ("clamped" plate)

(7) $\boxed{\psi = \partial_\nu \psi = 0 \quad \text{on } \gamma}$

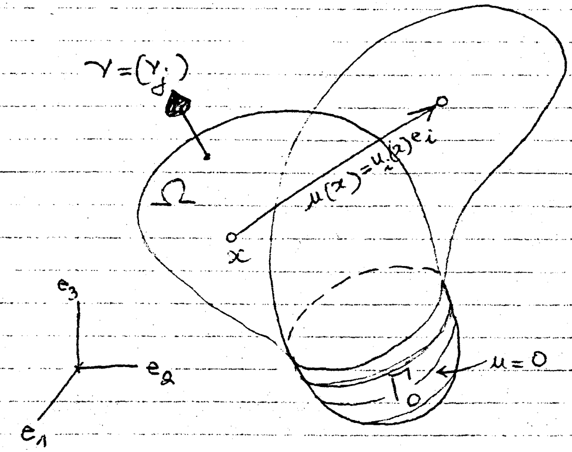
If (6) is acceptable, (7) is much more questionable, as we shall show. We hope also to clarify this point.

Remark. It is perfectly admissible that we do not introduce an Airy function; then we obtain 2d-models in $(u_3, \sigma^0_{\alpha\beta})$ or in (u_i), as we shall do here.

In the following work, we answer in particular a question raised by C. TRUESDEL. It seems that no justification of nonlinear plate models existed so far! (even with a priori assumptions).

(1) M.S. BERGER "Nonlinearity and Functional Analysis", Academic Press, 1977.

2. THE THREE-DIMENSIONAL NONLINEAR GENERAL MODEL [1]



$$\boxed{\bar{\gamma}_{ij}(v) = \gamma_{ij}(v) + \frac{1}{2}\partial_i u_\ell \, \partial_j u_\ell}$$

$$\Gamma_1 = \Gamma - \Gamma_0 \quad , \quad \Gamma = \partial\Omega$$

2.1. • The model. It corresponds to the energy (compare with (1)):

$$(8) \qquad \boxed{\bar{J}(v) = \frac{1}{2}\int_\Omega (A^{-1}\bar{\gamma}(v))_{ij}\, \bar{\gamma}_{ij}(v) - \left\{\int_\Omega f_i v_i + \int_{\Gamma_1} g_i v_i\right\}}$$

(functional space will be defined later). To write the equivalent system[2], it is convenient to introduce right now the unknowns σ_{ij} s.t. $(A\sigma)_{ij} = \bar{\gamma}_{ij}(u)$; $\sigma = (\sigma_{ij})$ is the (second) Piola-Kirchoff stress tensor.

$$(9) \qquad \begin{array}{l} (A\sigma)_{ij} = \bar{\gamma}_{ij}(u) \\ -\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i \\ u = 0 \text{ on } \Gamma_0 \\ (\sigma_{ij} + \sigma_{kj}\partial_k u_i)\nu_j = g_i \text{ on } \Gamma_1 \end{array}$$

← (linear stress-strain relation) but "full" strain tensor $\bar{\gamma}$

← Cauchy's law expressed in the reference configuration whence "large displacement" model

(2) As follows from applications of Green's formula.

(1) cf. C. TRUESDELL and W. NOLL: The Nonlinear Field Theories of Mechanics, in Handbuch der Physik, Vol. III/3, Springer, Berlin, 1965.

The *linear* stress - strain relation corresponds to an energy of the form (8). It can be shown(') that in a *general* energy

$$J(v) = \int_\Omega F(\gamma(v)),$$

this corresponds to the first term in the Taylor expansion of F (around $\gamma = 0$), whence our model corresponds to "*small*" *strains* γ.

Remark. Whereas in the linear case, the energy was *quadratic*, here we have *tri-* and *quadri- linear* terms in J.

2.2. • *Choice of function spaces for a variational formulation of (9).* We multiply eqns in (9) by test functions and integrate by parts. Formally:

$(A\sigma)_{ij} = \bar{\gamma}_{ij}(u) \Leftrightarrow$

$\left.\begin{array}{l} -\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i \\ (\sigma_{ij} + \sigma_{kj}\partial_k u_i)\nu_j = g_i \end{array}\right\} \Leftrightarrow$

($u = 0$ contained in def. of V)

(10)
$$\boxed{\begin{aligned} &\forall \tau \in \Sigma, \int_\Omega (A\sigma)_{ij}\tau_{ij} - \int_\Omega \tau_{ij}\gamma_{ij}(u) - \frac{1}{2}\int_\Omega \tau_{ij}\partial_i u_\ell \partial_j u_\ell = 0, \\ &\forall v \in V, \int_\Omega \sigma_{ij}\gamma_{ij}(v) + \int_\Omega \underbrace{\sigma_{ij}}_{\in L^2} \underbrace{\partial_i u_\ell}_{\in L^4} \underbrace{\partial_j v_\ell}_{\in L^4} = \int_\Omega f_i v_i + \int_{\Gamma_1} g_i v_i, \end{aligned}}$$

(11)
$$\boxed{\begin{aligned} &V = \{v = (v_i) \in (W^{1,4}(\Omega))^3 \,;\, v = 0 \text{ on } \Gamma_0\}, \\ &\Sigma = \{\tau = (\tau_{ij}) \in (L^2(\Omega))^9 \,;\, \tau_{ij} = \tau_{ji}\}. \end{aligned}}$$

(') cf. e.g. R. VALID: "*La Mécanique des Milieux Continus et le Calcul des Structures*", Eyrolles, Paris, 1977.

2.3. • Existence of a solution. We only obtain a partial result for:

– the pure Dirichlet problem ($u = 0$ on $\Gamma_0 = \Gamma$)(1),

– sufficiently small applied forces.

Principle of the proof: We eliminate the unknowns σ_{ij} and after integrating by parts ($\int_\Omega (\ldots)\, \partial_j v_i = \int_\Omega (\mathcal{A}u) v_i$) we obtain:

$$\mathcal{A}(u) = f \quad \text{(in the distribution sense at least)}$$

with (writing now $(A^{-1}Y)_{ij} = a_{ijkl} Y_{kl}$) (2)

$$(\mathcal{A}(u))_i = -\partial_j \Big(a_{ijkl}\, \gamma_{kl}(u) + \frac{1}{2} a_{ijkl}\, \partial_k u_m\, \partial_l u_m + a_{ljki}\, \gamma_{ki}(u)\, \partial_l u_i + \frac{1}{2} a_{ljkp} \underbrace{\overbrace{\partial_k u_m}^{\in W^{1,4}} \overbrace{\partial_p u_m}^{\in W^{1,4}} \overbrace{\partial_l u_i}^{\in W^{1,4}}}_{\in W^{1,4}} \Big).$$

Because $W^{1,4}(\Omega)$, $\Omega \subset \mathbb{R}^3$, is an algebra (cf. ADAMS' book), $\mathcal{A}$ maps $(W^{2,4}(\Omega))^3$ into $(L^4(\Omega))^3$ and is of class $\mathcal{C}^1$ (sum of k-linear continuous mappings; $W^{1,4}(\Omega)$ is an algebra).

Now $\mathcal{A}'(0)$ is nothing but the linear elasticity system!

Consequently, if we can prove that

$$\mathcal{A}'(0) : (W^{2,4}(\Omega))^3 \to (L^4(\Omega))^3$$

is an isomorphism, existence around the origin will follow from the implicit function theorem.

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(1) The extension to $u = u_0$ on $\Gamma_0 = \Gamma$ is possible.

(2) It is simply shorter to use here the coefficients a_{ijkl} rather than the Lamé's constants introduced p. 1.

In other words, we need a <u>regularity result</u>: for all $f \in L^4(\Omega)$, there exists a solution in $W^{2,4}(\Omega)$. This follows from:

(i) $H^2(\Omega)$-regularity for $f \in L^2(\Omega)$ for the elasticity system (cf. NEČAS' book, p. 260).

(ii) the <u>index</u> of the mapping

$$\mathcal{A}'(0) : (W^{2,p}(\Omega))^3 \to (L^p(\Omega))^3$$

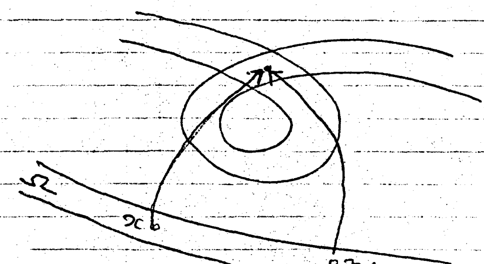
is <u>independent</u> of $p \in]1, \infty[$ (1) ($\mathcal{A}'(0)$ is injective)

<u>Remark</u>. Contrary to a common belief, this does <u>not</u> follow from AGMON-DOUGLIS-NIRENBERG; who rather prove: <u>If</u> we have the $W^{2,p}$-regularity, then $f \in W^{m,p} \Rightarrow u \in W^{m+2,p}$ for any $m \geq 1$.

2.4. <u>1-1 character of the mapping</u>

$$\phi : x \in \Omega \to \phi(x) = x + u(x).$$

Of course, it is desirable to avoid the following situation:



(1) cf. G. GEYMONAT: Sui problemi ai limiti per i sistemi lineari ellittici, Ann. Mat. Pura Appl. LXIX (1965), 207-284.

One has

Jacobian of ϕ at $x = \boxed{J_\phi(x) = \det\left(I + (\partial_j u_i)\right)}$,

hence if $|u|_{1,\infty,\Omega}$ is <u>small enough</u>,

$$\forall x \in \bar\Omega,\ J_\phi(x) \neq 0 .$$

But this follows from the previous result and

$$\boxed{W^{2,4}(\Omega) \hookrightarrow \mathcal{C}^1(\bar\Omega)}$$

Using (1), we know that

$$\left.\begin{array}{r} \phi : \mathcal{O} \supset \Omega \to \underset{\sim}{\mathbb{R}}^3 \text{ of class } C^1 \\ \forall x \in \bar\Omega,\ J_\phi(x) \neq 0 \ (^2) \\ \phi|_\Gamma \text{ is } 1\text{-}1 \end{array}\right\} \Rightarrow \phi : \bar\Omega \to \mathbb{R}^3 \text{ is } 1\text{-}1 .$$

whence the conclusion follows.

<u>Remark</u> (in passing): Application to isoparametric f.e. !

2.5. ● <u>Open problems</u>. (i) Existence by other means (elsewhere than around 0). Results of Ball ?

(ii) Even with the implicit function thm, corresponding regularity result for the 3d-clamped plate problem ? (only hope is because cylindrical domain; otherwise even H^2-regularity does not hold for Dirichlet <u>and</u> Neumann b.c.).

(iii) <u>Numerical</u> analysis of f.e.m. for this 3d-problem ? Any reference ?

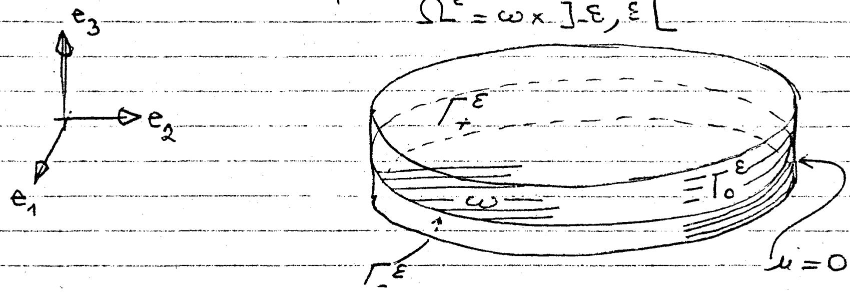
(2) This condition may be relaxed to $\Omega - \{$finite set$\}$ and $\Gamma - \{$non-empty

(1) G.H. MEISTERS and C. OLECH, "Locally one-to-one mappings and a classical theorem on Schlicht functions, Duke Math. J. <u>30</u> (1963), 63-80.

3. THE PLATE PROBLEM; APPLICATION OF THE ASYMPTOTIC EXPANSION METHOD

3.1. The 3d-problem.

$$\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$$



Applied forces:

$f = (f_i)$ ~~in~~ in Ω^ε

$g = (g_i)$ on Γ_0^ε

$u = 0$ on Γ_0^ε ("clamped" plate)

$$\boxed{\begin{aligned}\Sigma^\varepsilon &= \{\tau = (\tau_{ij}) \in (L^2(\Omega^\varepsilon))^9 \,;\, \tau_{ij} = \tau_{ji}\}.\\ V^\varepsilon &= \{v = (v_i) \in (W^{1,4}(\Omega^\varepsilon))^3 \,;\, v = 0 \text{ on } \Gamma_0^\varepsilon\},\end{aligned}}$$

$$\forall \tau \in \Sigma^\varepsilon, \quad \int_{\Omega^\varepsilon} (A\sigma)_{ij}\tau_{ij} - \int_{\Omega^\varepsilon} \tau_{ij}\gamma_{ij}(u) - \frac{1}{2}\int_{\Omega^\varepsilon} \tau_{ij}\partial_i u_\ell \partial_j u_\ell = 0,$$

$$\forall v \in V^\varepsilon, \quad \int_{\Omega^\varepsilon} \sigma_{ij}\gamma_{ij}(v) + \int_{\Omega^\varepsilon} \sigma_{ij}\partial_i u_\ell \partial_j v_\ell = \int_{\Omega^\varepsilon} f_i v_i + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i v_i$$

Remark. The functions f_i and g_i are assumed smooth enough for all subsequent purpo

3.2. Transformation into a problem posed over a domain independent of ε.

Objective: To make as simple as possible the dependence on ε. We let

$$\Omega = \omega \times]-1, 1[\; = \Omega^1$$

$$\Gamma_0 = \Gamma_0^1, \quad \Gamma_\pm = \Gamma_\pm^1,$$

$$V = V^1, \; \Sigma = \Sigma^1.$$

We make the following changes of variables and functions

$$X = (x_1, x_2, x_3) \in \overline{\Omega} \to X^\varepsilon = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega}^\varepsilon$$

(12)
$$\boxed{\begin{aligned} &\sigma_{\alpha\beta}(X^\varepsilon) = \sigma^\varepsilon_{\alpha\beta}(X), \ \sigma_{\alpha 3}(X^\varepsilon) = \varepsilon \sigma^\varepsilon_{\alpha 3}(X), \ \sigma_{33}(X^\varepsilon) = \varepsilon^2 \sigma^\varepsilon_{33}(X) \\ &v_\alpha(X^\varepsilon) = v^\varepsilon_\alpha(X), \quad v_3(X^\varepsilon) = \varepsilon^{-1} v^\varepsilon_3(X), \end{aligned}}$$

(as a result: $\varepsilon \int_\Omega \sigma^\varepsilon_{ij} \gamma_{ij}(v^\varepsilon) = \int_{\Omega^\varepsilon} \sigma_{ij} \gamma_{ij}(v)$),

(13)
$$\boxed{\begin{aligned} &f_\alpha(X^\varepsilon) = \varepsilon^2 f^\varepsilon_\alpha(X), \quad f_3(X^\varepsilon) = \varepsilon^3 f^\varepsilon_3(X), \\ &g_\alpha(X^\varepsilon) = \varepsilon^3 g^\varepsilon_\alpha(X), \quad g_3(X^\varepsilon) = \varepsilon^4 g^\varepsilon_3(X). \end{aligned}}$$

(as a result: $\int_{\Omega^\varepsilon} f_i v_i + \int_{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-} g_i v_i = \varepsilon^3 \left(\int_\Omega f^\varepsilon_i v^\varepsilon_i + \int_{\Gamma_+ \cup \Gamma_-} g^\varepsilon_i v^\varepsilon_i \right)$).

Proposition The element $(\sigma^\varepsilon, u^\varepsilon) \in \Sigma \times V$ obtained from $(\sigma, u) \in \Sigma^\varepsilon \times V^\varepsilon$ through (12), satisfies:

(14)
$$\forall \tau \in \Sigma, \ \mathcal{A}_0(\sigma^\varepsilon, \tau) + \varepsilon^2 \mathcal{A}_2(\sigma^\varepsilon, \tau) + \varepsilon^4 \mathcal{A}_4(\sigma^\varepsilon, \tau) + \\ + \mathcal{B}(\tau, u^\varepsilon) + \mathcal{C}_0(\tau, u^\varepsilon, u^\varepsilon) + \varepsilon^{-2} \mathcal{C}_{-2}(\tau, u^\varepsilon, u^\varepsilon) = 0,$$

(15)
$$\forall v \in V, \quad \mathcal{B}(\sigma^\varepsilon, v) + 2\mathcal{C}_0(\sigma^\varepsilon, u^\varepsilon, v) + 2\varepsilon^{-2} \mathcal{C}_{-2}(\sigma^\varepsilon, u^\varepsilon, v) = \varepsilon^2 \mathcal{F}(v)$$

where in particular (we record only the expressions useful in the sequel):

ue that all
es in $\mathcal{A}_0$ are in $\{1,2\}$)

(16)
$$\boxed{\begin{aligned} &\mathcal{A}_0(\sigma, \tau) = \int_\Omega \left\{ \frac{(1+\nu)}{E} \sigma_{\alpha\beta} - \frac{\nu}{E} \sigma_{\mu\mu} \delta_{\alpha\beta} \right\} \tau_{\alpha\beta}, \\ &\mathcal{B}(\tau, v) = -\int_\Omega \tau_{ij} \gamma_{ij}(v); \quad \mathcal{C}_{-2}(\tau, u, v) = -\frac{1}{2} \int_\Omega \tau_{ij} \partial_i u_3 \partial_j v_3, \\ &\mathcal{F}(v) = \int_\Omega f^\varepsilon_i v_i + \int_{\Gamma_+ \cup \Gamma_-} g^\varepsilon_i v_i. \end{aligned}}$$

3.3. Formal expansion of $(\sigma^\varepsilon, u^\varepsilon)$

Equations (14)-(15) suggest that we let

$$(17) \qquad \boxed{(\sigma^\varepsilon, u^\varepsilon) = \varepsilon^2(\sigma^2, u^2) + \varepsilon^3(\sigma^3, u^3) + \cdots} \; .$$

Then we plug this formal expansion into (14)-(15) and we equate to zero the factors of the successive powers of ε. In this fashion, we obtain

i) equations to be satisfied by (σ^2, u^2),

ii) recurrence relations satisfied by the next terms.

Remarks. – At this stage this is completely formal; nothing guarantees that such (σ^p, u^p) exist in $\Sigma \times V$ or even in a larger space.

– If we had started by ε^p, $p \le 1$, then the resulting eqns for (σ^p, u^p) correspond to $u_3^p = 0$ (an unwanted property for what is supposed to be an approximation of the 3d-problem. Besides, it does not "contain" the linear case. □

By inspection we find that (σ^2, u^2) should satisfy

Add^l terms wrt the linear case

$$(18) \qquad \forall \tau \in \Sigma, \quad \mathcal{A}_0(\sigma^2, \tau) + \mathcal{B}(\tau, u^2) + \overbrace{\mathcal{C}_{-2}(\tau, u^2, u^2)} = 0,$$

$$(19) \qquad \forall v \in V, \quad \mathcal{B}(\sigma^2, v) + \overbrace{2\,\mathcal{C}_{-2}(\sigma^2, u^2, v)} = \mathcal{F}(v).$$

(consider the factors of ε^2 = the smallest power of ε).

4. MAIN RESULTS

• Theorem. If the forces f_α, g_α are sufficiently small (1), problem (18)-(19) has (at least) one solution in the space $\Sigma \times V$, which coincides with the solution of a known nonlinear 2d-plate model.

Idea of the proof. From now on, we let $(\sigma^2, u^2) = (\sigma, u)$ for notational brevity.

Step 1. (u_i) is a Kirchoff-Love displacement field.

Let us write eqns (18) for $\tau = \begin{pmatrix} 0 & 0 & \tau_{13} \\ 0 & 0 & \tau_{23} \\ \tau_{31} & \tau_{23} & 0 \end{pmatrix}$ and $\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}$:

$$\forall \tau_{\alpha 3} \in L^2(\Omega), \quad \int_\Omega \tau_{\alpha 3} \left(\partial_\alpha u_3 + \partial_3 u_\alpha + \partial_\alpha u_3 \partial_3 u_3 \right) = 0$$

$$\forall \tau_{33} \in L^2(\Omega), \quad \int_\Omega \tau_{33} \left(\partial_3 u_3 + \frac{1}{2} (\partial_3 u_3)^2 \right) = 0,$$

whence

$$\begin{cases} \partial_\alpha u_3 + \partial_3 u_\alpha + \partial_\alpha u_3 \partial_3 u_3 = 0, \\ \partial_3 u_3 + \frac{1}{2} (\partial_3 u_3)^2 = 0 \longrightarrow \text{either } \partial_3 u_3 = 0 \text{ or } \partial_3 u_3 = -2. \end{cases}$$

extra terms wrt the linear case.

To circumvent the ambiguity, let us henceforth restrict ourselves to those solutions u_3 which are in $W^{2,4}(\Omega) \hookrightarrow \mathcal{C}^1(\bar{\Omega})$ (2), whence $\partial_3 u_3 = -2$ ruled out ($u_3 = 0$ on Γ_0).

$$\partial_3 u_3 = 0 \Rightarrow \partial_\alpha u_3 + \partial_3 u_\alpha = 0 \Rightarrow \cancel{\partial_{\alpha 3} u_3} + \partial_{33} u_\alpha = 0$$

$$\Rightarrow \exists\, u_\alpha^0, u_\alpha^1 \in W_0^{1,4}(\omega) \text{ s.t. } u_\alpha = u_\alpha^0 + x_3 u_\alpha^1 .$$

$\therefore\ \partial_\alpha u_3 = -\partial_3 u_\alpha = -u_\alpha^1 \quad \therefore\ u_3 \in W_0^{2,4}(\omega)$ (u_3 ind^t of x_3, in $W^{2,4}(\Omega)$, $= 0$ on Γ_0, and $\partial_\alpha u_3 \in W_0^{1,4}(\omega)$)

(1) Therefore, no restriction is imposed upon the functions f_3, g_3.
(2) This is a posteriori justified by the fact that we can find ~~[illegible]~~ solutions possessing this regularity.

contains the b.c. for u_3

To sum up:

$$\boxed{\begin{array}{l} u_3 \text{ is independent of } x_3 \text{ and is in } W_0^{2,4}(\omega) \\ \exists u_\alpha^0 \in W_0^{1,4}(\omega), \quad u_\alpha = u_\alpha^0 - x_3 \partial_\alpha u_3 . \end{array}}$$

Remark. This is also the first step towards the transformation into a 4th-order problem, since $u_3 \in W_0^{2,4}(\omega)$. □

Remark. In the linear case, no need to assume u_3 is in $H^2(\Omega)$; it is automatically found. □

Step 2. Computation of the functions (u_α^0, u_3).

We let successively (all other components are zero)

$$\begin{cases} \tau_{\alpha\beta} = \tau_{\alpha\beta}^0 \in L^2(\omega) \text{ in } (18) \\ v_\alpha = v_\alpha^0 \in W_0^{1,4}(\omega) \text{ in } (19) \end{cases}$$

$$\begin{cases} \tau_{\alpha\beta} = x_3 \hat{\tau}_{\alpha\beta} \in L^2(\omega) \text{ in } (18) \\ \left.\begin{array}{l} v_\alpha = x_3 \partial_\alpha v \\ v_3 = v \end{array}\right\} v \in W_0^{2,4}(\omega) \text{ in } (19). \end{cases}$$

(if (18) and (19) are to be satisfied, then they should be satisfied in particular by such fcns; a remarkable fact is that it is an iff cond.)

Then after elimination of the other unknowns, we find a 2d-problem of the form: Find $(u_1^0, u_2^0, u_3) \in (W_0^{1,4}(\omega))^2 \times W_0^{2,4}(\omega)$ s.t.

$$(20) \quad \begin{cases} \forall v_\alpha^0 \in W_0^{1,4}(\omega), \ldots \\ \forall v \in W_0^{2,4}(\omega), \ldots \end{cases}$$

For simplicity only, assume $f_\alpha = g_\alpha = 0$. Then (20) is equivalent to (after returning to the set Ω^ε):

$\neq 0$ in the linear case

$$\frac{2E\varepsilon^3}{3(1-\nu^2)} \Delta^2 u_3 = \varepsilon \sum_{\alpha,\beta=1} \sigma_{\alpha\beta}^0 \partial_{\alpha\beta} u_3 + \left(g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3 \, dx_3\right)$$

$$\partial_\alpha \sigma_{\alpha\beta}^0 = 0, \text{ where } \cancel{\sigma_{\alpha\beta}^0 = \frac{2E}{(1-\nu^2)} \varepsilon (\ldots)}$$

$$u_\alpha^0 = 0 \text{ on } \gamma, \quad u_3 = \partial_\nu u_3 = 0 \text{ on } \gamma$$

equivalent to (after returning to the set Ω^ε):

$$\text{(21)}\quad \boxed{\begin{aligned} &\text{(a)} && \frac{2E\varepsilon^3}{3(1-\nu^2)}\Delta^2 u_3 = \varepsilon\,\sigma^0_{\alpha\beta}\partial_{\alpha\beta}u_3 + \left(g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3\,dx_3\right)\\ &\text{(b)} && \partial_\alpha \sigma^0_{\alpha\beta} = 0,\\ &\text{(c)} && u^0_\alpha = \text{on } \gamma, \quad u_3 = \partial_\nu u_3 = 0 \text{ on } \gamma, \end{aligned}}$$

where

$$\text{(21')}\quad \boxed{\begin{aligned} \sigma^0_{11} &= \frac{2E}{(1-\nu^2)}\left\{\partial_1 u^0_1 + \tfrac{1}{2}(\partial_1 u_3)^2 + \nu\left(\partial_2 u^0_2 + \tfrac{1}{2}(\partial_2 u_3)^2\right)\right\},\\ \sigma^0_{12} &= \frac{2E}{(1+\nu)}\left\{\gamma_{12}(u^0) + \partial_1 u_3\,\partial_2 u_3\right\},\\ \sigma^0_{22} &= \frac{2E}{(1-\nu^2)}\left\{\partial_2 u^0_2 + \tfrac{1}{2}(\partial_2 u_3)^2 + \nu\left(\partial_1 u^0_1 + \tfrac{1}{2}(\partial_1 u_3)^2\right)\right\}. \end{aligned}}$$

Remarks. The notation $\sigma^0_{\alpha\beta}$ is justified because $\sigma^0_{\alpha\beta} = \sigma_{\alpha\beta}(\cdot,\cdot,0)$[1]. Likewise, observe that $u^0_\alpha = u_\alpha(\cdot,\cdot,0)$. □

UCIAL CONCLUSION: We have therefore obtained a known nonlinear 2d-model for plates (cf. e.g. the books of STOKER and WOINOWSKY-KRIEGER). Notice in particular that the boundary conditions (which involve the functions u^0_α and u_3) have been found without any ambiguity. ■

(1) ~~cf. Ste~~ this can be seen only in Step 3.

Step 3. If the norms $\|g_\alpha\|_{L^2(\Gamma_+ \cup \Gamma_-)}$ and $\|f_\alpha\|_{L^2(\Omega)}$ are small enough (1), problem (20) (= (21) if $f_\alpha = g_\alpha = 0$) has at least solution, which has the following regularity:

$$u = (u_1^0, u_2^0, u_3) \in \left(W_0^{1,4}(\omega) \cap W^{3,4}(\omega)\right)^2 \times \left(W_0^{2,4}(\omega) \cap W^{4,4}(\omega)\right).$$

Principle: Eqns (20) assert that $j'(u)v = 0$, for an appropriate functional j, already defined over the space $W = (H_0^1(\omega))^2 \times H_0^2(\omega)$. On this space, $j \to \infty$ as $\|v\|_W \to \infty$ (0). Next, although j is not convex, we show it is weakly lower semi-continuous on W (in particular because the injection $H_0^2(\omega) \hookrightarrow W^{1,4}(\omega)$ is compact).

The asserted regularity follows from an argument similar to that used by (2).

Step 4. Computation of the stresses: All the functions σ_{ij} are given by explicit formulas involving the functions u_α^0 and u_i.

Then it is an easy matter to check that we have indeed obtained a solution to (18)-(19). □

Conclusion: Without any a priori assumption, either of a mechanical or geometrical nature, we have found a known nonlinear 2d-plate model.

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(0) This is where we need that f_α, g_α be small. Also, this property would not be true on the original space.

(1) We now return to the general case ($f_\alpha \neq 0$, $g_\alpha \neq 0$).

(2) LIONS, J.L.: Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod, Paris, 1969.

5. INTRODUCTION OF THE AIRY STRESS FUNCTION

Let us return to the case where $f_\alpha = g_\alpha = 0$; cf. eqns (21).

<u>Lemma 1</u>. $\left.\begin{array}{l} \sigma^0_{\alpha\beta} \in W^{2,4}(\omega) \ (^1) \\ \partial_\alpha \sigma^0_{\alpha\beta} = 0 \\ \sigma^0_{12} = \sigma^0_{21} \end{array}\right\} \Rightarrow$ $\exists!\, \phi \in W^{4,4}(\omega)/P_1(\omega)$ (²) s.t. $\boxed{\partial_{11}\phi = \sigma^0_{22},\ \partial_{12}\phi = -\sigma^0_{12} = -\sigma^0_{21},\ \partial_{22}\phi = \sigma^0_{11}.}$ (³)

<u>Proof</u>. Relies essentially on Poincaré's theorem properly extended to Sobolev's spaces. □

Equations (21a) then become

$$\boxed{\frac{2E\varepsilon^3}{3(1-\nu^2)}\Delta^2 u_3 = 2\varepsilon\,[\phi, u_3] + \left(g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3\, dx_3\right)} \tag{23}$$

and we still have, by (21c):

$$\boxed{u_3 = \partial_\nu u_3 = 0 \text{ on } \gamma.} \tag{24}$$

On the other hand, a straightforward computation shows that

$$\boxed{\Delta^2\phi = -E\,[u_3, u_3]} \tag{25}$$

<u>Conclusion</u>: (23) and (25) are the <u>von Kármán equations</u>; We have (in (24)) b.c. for u_3.

<u>It remains to find ~~the~~ appropriate b.c. for ϕ.</u>

~~Preliminary~~

(¹) As follows from Step 4 of the previous theorem.
(²) $P_1(\omega)$ = space of pol. of degree ≤ 1 over ω.
(³) ϕ is called the AIRY stress function.

Let ϕ_0 be the (unique) solution of

$$(26)\qquad \boxed{\begin{array}{l} \Delta^2\phi_0 = 0 \quad \text{in } \omega \\ \left.\begin{array}{l}\phi_0 = \phi \\ \partial_\nu \phi_0 = \partial_\nu \phi\end{array}\right\} \text{ on } \gamma \end{array}} \quad (1)$$

Then the functions u_3 and

$$\boxed{\psi = \phi - \phi_0}$$

satisfy

$$(27)\qquad \boxed{\begin{array}{l} \dfrac{2E\varepsilon^3}{3(1-\nu^2)}\Delta^2 u_3 = 2\varepsilon[\psi, u_3] + 2\varepsilon[\phi_0, u_3] + \left(g_3^+ + g_3^- + \int_{-\varepsilon}^{\varepsilon} f_3\, dx_3\right) \\ \Delta^2\psi = -E[u_3, u_3] \\ u_3 = \partial_\nu u_3 = 0 \quad \text{on } \gamma \\ \psi = \partial_\nu \psi = 0 \quad \text{on } \gamma \end{array}}$$

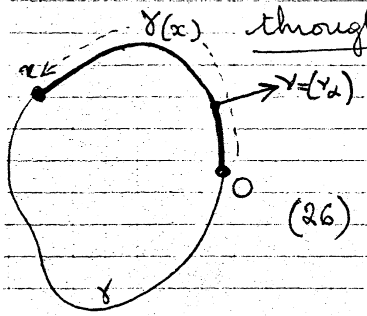
<u>Conclusion</u>: If we want to impose the b.c. $\psi = \partial_\nu\psi = 0$ on γ, this is <u>at the expense of adding the term</u> $[\phi_0, u_3]$ <u>in the first equations.</u> <u>There is no reason to expect ϕ_0 to vanish.</u>

Let us now examine how to compute $\phi, \partial_\alpha\phi$ along γ. From Lemma 1, it seems that we can only compute the 2nd partial derivatives $\partial_{\alpha\beta}\phi$ {from the knowledge of the $\sigma^0_{\alpha\beta}$. However we have:

---

(1) Once we have solved our 2d-problem as in Sect. 4, ~~it~~ the functions ϕ is known (up to a pol. of degree 1) by Lemma 1,

Lemma 2. Assume wlg that $O \in \gamma$. We define ϕ uniquely by specifying that $\phi(0) = \partial_1\phi(0) = \partial_2\phi(0)$. Then one can compute the functions $\phi, \partial_1\phi, \partial_2\phi$ along γ as functions of the ~~4~~ quantities $\overset{\circ}{\sigma}_{\alpha\beta}$, through the formulas:



(26)
$$\boxed{\begin{aligned}
\partial_1\phi(x) &= -\int_{\gamma(x)} h_2 \\
\partial_2\phi(x) &= \int_{\gamma(x)} h_1 \\
\phi(x) &= \int_{\gamma(x)} (x_1 h_2 - x_2 h_1) - x_1 \int_{\gamma(x)} h_2 + x_2 \int_{\gamma(x)} h_1
\end{aligned}}$$

where

(27)
$$\boxed{\begin{aligned}
h_1 &= \overset{\circ}{\sigma}_{11}\nu_1 + \overset{\circ}{\sigma}_{21}\nu_2 \\
h_2 &= \overset{\circ}{\sigma}_{12}\nu_1 + \overset{\circ}{\sigma}_{22}\nu_2
\end{aligned}}$$

Conclusion: This suggests that the original 3d-problem be defined with the following b.c. on Γ_0^ε:

(28)
$$\boxed{\left.\begin{aligned}
u_3 &= 0 \\
\sigma_{11}\nu_1 + \sigma_{21}\nu_2 &= h_1 \\
\sigma_{12}\nu_1 + \sigma_{22}\nu_2 &= h_2
\end{aligned}\right\} \text{ on } \Gamma_0^\varepsilon}$$

where h_1, h_2 are given functions. In the linear case at least, this is a perfectly admissible set of b.c. provided the applied forces satisfy a suitable compatibility condition (cf. e.g. DUVAUT & LIONS).

Assuming we can do this (the details remain to be checked)(*), let us examine various special cases. For simplicity, assume we started with

$$\begin{cases} \Delta^2 u_3 = [\psi, u_3] + [\Phi_0, u_3] + f & \text{in } \omega \\ \Delta^2 \psi = -[u_3, u_3] & \text{in } \omega \\ u_3 = \partial_\nu u_3 = 0 & \text{on } \gamma \\ \psi = \partial_\nu \psi = 0 & \text{on } \gamma \end{cases}$$

Uniform pressure, or traction, along γ:

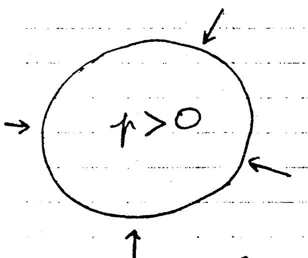
$$\sigma^0_{\alpha\beta} = p\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad p \in \mathbb{R}$$

The unique solution of problem (26) is seen to be (apply Lemma 2):

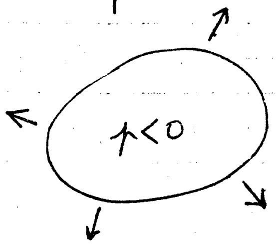
$$\Phi_0 = \left(p\, \frac{x_1^2 + x_2^2}{2}\right).$$

Whence the equation

$$\Delta^2 u_3 = [\psi, u_3] + p\,\Delta u_3 + f$$



$p>0 \longrightarrow$ bifurcation (around 0 when $f=0$; cf the book of BERGER)



$p<0 \longrightarrow$ membrane theory, $pu_3 \to$ sol of $\boxed{-\Delta u = f}$

(*) In particular, it seems that we shall not obtain the boundary condition $\partial_\nu u_3 = 0$ on γ. Besides, there remain some problems as regards the nonlinearity.

6. FINAL REMARKS

Open problems. 1) Apply all this to evolution problems

2) Convergence analysis in the nonlinear case.

3) Existence of a 3d-solution around a 2d-solution?

etc...